Effect of spatially correlated noise on coherence resonance in a network of excitable cells

Okyu Kwon,* Hang-Hyun Jo, and Hie-Tae Moon
Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea
(Received 10 June 2005; published 22 December 2005)

We study the effect of spatially correlated noise on coherence resonance (CR) in a Watts-Strogatz small-world network of Fitz Hugh–Nagumo neurons, where the noise correlation decays exponentially with distance between neurons. It is found that CR is considerably improved just by a small fraction of long-range connections for an intermediate coupling strength. For other coupling strengths, an abrupt change in CR occurs following the drastic fracture of the clustered structures in the network. Our study shows that spatially correlated noise plays a significant role in the phenomenon of CR reinforcing the role of the clustered structure of the system.

DOI: 10.1103/PhysRevE.72.066121 PACS number(s): 84.35.+i, 89.75.Hc, 05.45.−a, 05.45.Xt

The response of nonlinear systems to noise has attracted a large amount of attention. Especially stochastic resonance (SR) has been studied very extensively during the last two decades due to a number of applications in many fields, from physical to biological systems [1–6]. The main result of SR, which is somewhat counterintuitive, shows that noise at a proper strength optimizes the response of a nonlinear system to a subthreshold periodic signal. An optimal strength of noise can induce the most coherent motion in the system. SR-like behavior of the coherent motion can be induced purely by noise even in the absence of an external periodic signal for systems with limit cycles or self-sustained oscillations at close to the bifurcation point [7]. This phenomenon has been called coherence resonance (CR) or autonomous SR. In general, various excitable systems such as the Fitz Hugh–Nagumo model [8], the Plant model, the Hindermarsh-Rose model [9], and the Hodgkin-Huxley model [10] exhibit such noise-induced coherent motion.

Recently, SR and CR in coupled or extended systems have become an interesting issue, and some new features have been demonstrated, namely the noise-enhanced phase synchronization [11], the noise-induced spatiotemporal pattern formation [12], and the noise-enhanced wave propagation [13,14]. Also, the phenomena called array enhanced stochastic resonance (AESR) and array enhanced coherence resonance (AECR) [15–19] have drawn interests among researchers in recent years. It is now understood that in spatially extended systems (i) the topology of connecting structure and (ii) noise correlation among the elements are the significant ingredients on the collective behavior of the systems. Actually, the connecting topology of a variety of extended systems can be described by complex networks [20]. Especially many biological neural networks present clear clustered structure and sparsely long-range random connectivity [21]. On the other hand, spatially correlated noise has been considered relevant for biological systems [22]. However, most previous studies have not dealt with these factors together [17–19,23]. In this study, we plan to add the spatially correlated noise in the system of excitable cells and to study the corresponding dynamics of CR.

As a model, we consider a system of coupled excitable cells on "small-world" network, introduced by Watts and Strogatz [24] in the presence of a spatially correlated noise. Each cell is a Fitz Hugh–Nagumo (FHN) neuron which is a simple but representative model of excitable neuron [25]. N neurons in a ring lattice are diffusively coupled as follows:

$$\epsilon \dot{x}_i = x_i - \frac{x_i^3}{3} - y_i + \sum_j g_{ij}(x_j - x_i), \tag{1}$$

$$\dot{y}_i = x_i + a + \xi_i, \tag{2}$$

where x_i is the fast voltage variable and y_i is the slow recovery variable of ith neuron. ϵ and a are a time scale and a bifurcation parameter, respectively. We fix $\epsilon=0.01$ and $a=1.03$ for all $N=101$ neurons. For $|a|>1$, a single FHN neuron has only a stable fixed point. While for $|a|<1$, a limit cycle occurs. g_{ij} is a coupling strength between two neurons i and j. If connected, they have the coupling strength $g_{ij}=g$, otherwise $g_{ij}=0$. The connectivity pattern can vary with parameter p, which measures the network randomness. ξ_i is spatially correlated noise with intensity D.

The spatially correlated noise ξ is generated by summing N Gaussian white noises ζ with correlation function C

$$\xi_i = \frac{1}{\sqrt{\sum_{k\in\Lambda} C_k^2}} \sum_{k\in\Lambda} \zeta_{i+k} C_k, \tag{3}$$

$$\Lambda = \{-4\lambda, \ldots, -2, -1, 0, 1, 2, \ldots, 4\lambda\}, \tag{4}$$

where ζ_i is a Gaussian white noise with zero mean and correlation given by $\langle \zeta_i(t)\zeta_j(t')\rangle = D\delta_{ij}\delta(t-t')$; D denoting the noise intensity. And the correlation function is defined by $C_k=\exp(-2k^2/\lambda^2)$. According to the above method, we can get the spatially correlated noise which obeys the correlation function with decay constant λ as follows:

$$\langle \xi_i(t)\xi_j(t')\rangle = D \exp\left(-\frac{|i-j|^2}{\lambda^2}\right)\delta(t-t'), \tag{5}$$

where i and j denote the spatial positions of neurons in the ring lattice. Consequently $|i-j|$ represents a distance along

*Electronic address: okyou@kaist.ac.kr

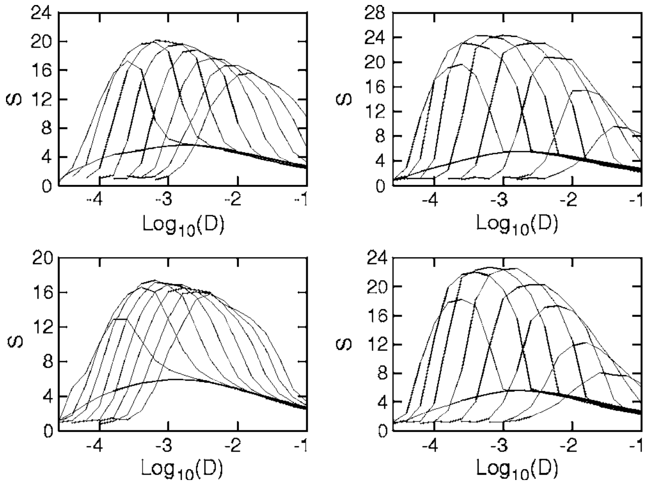

FIG. 1. The coherence factor S versus noise intensity D for several values of coupling strength on (a) regular network ($p=0$) and (b) completely random network ($p=1$) with spatial correlation length of noise $\lambda=0$. (c), (d) are the same curves for $p=0$ and $p=1$ with $\lambda=2$, respectively. Average number of neighbors $k=6$. Number of elements $N=101$. Coupling strength g varies from $g=10^{-2.5}$ to $g=10^{-0.5}$ with 0.25 step of exponent. Base curves represent $g=10^{-2.5}$. As coupling strength increases, the peak of S shifts to a stronger noise.

the ring lattice not the connecting topology. The systems are numerically integrated by the method of Fox *et al.* [26] with the time step $\Delta t=0.002$ t.u. (time units).

As a quantitative observable of a neuron showing CR, a temporal coherence is calculated by

$$S_i = \frac{\langle \tau_i \rangle_t}{\sqrt{\mathrm{var}(\tau_i)}}, \quad i = 1,2,\ldots,N, \tag{6}$$

where τ_i is the ensemble of time interval of interspike and S_i denotes temporal coherence factor of ith neuron. Here $\langle \cdot \rangle_t$ denotes average over time. The coherence factor S of the system is computed by averaging S_i over all N neurons. A larger S implies that the interspike intervals of neurons are more uniform.

A general feature of SR and CR is that there exists an optimal noise intensity at which the coherence factor is maximized. In the cases of AESR and AECR there exists an optimal coupling strength additionally [15,16,18]. Our numerical simulations verify the above results (see Fig. 1), that is, both the optimal noise intensity and the optimal coupling strength exist, moreover, regardless to the network randomness p and the correlation length λ of the noise. As coupling strength increases, noise-induced individual and coupling-induced mutual excitation enhances a coherent motion in the coupled system. However, since excitable elements spend most of their time in the rest state, a too strong coupling prevents excitation by noise. For this reason, a stronger noise is needed to overcoming stronger coupling to excite the neurons. In this case a global synchronization emerges due to strong coupling while the temporal coherence of the system is somewhat reduced due to strong noise. As seen in Fig. 1, the resonance curve shifts to the right side gradually and its peak rises up first and then drops down according to the coupling strength.

How do the connecting topology and the spatial correlation of noise influence the coupled excitable neurons? To focus on the question the effect of structural changes of connectivity on the spatial correlation of noise must be explained in advance. For a regular network ($p=0$), if λ is large enough, each neuron interacts only with the ones exposed to the correlated noise. As p increases each neuron is able to interact with a distant one which is exposed to an uncorrelated noise. Consequently for a proper length of λ, increasing p reduces the correlation of the noise and simultaneously increases the small-world effect between coupled neurons. However, when $\lambda=0$ every neuron is already exposed to a totally uncorrelated noise, i.e., local noise, so varying p does not alter the correlation of noise. In this case the variation of p influences the small-world effect only.

Now let us figure out the peculiar result according to p and λ. In Fig. 1 the maximum value of each curve is called a maximal coherence factor S_m and the noise intensity at S_m is called an optimal noise D_{opt}. If the coupling strength is not very strong, S_m takes a larger value for completely random network ($p=1$) than for regular network ($p=0$). On the other hand, S_m shows decline for the more random network as the coupling strength increases further. For large p, long-range connections amplify the effect of coupling for the entire range of g. Therefore, when the coupling is not very strong the long-range connections enhance further the coherent motion of the system. However, for the very strong coupling strength, long-range connections more reinforce the synchronization. This strong synchronization more prevents excitation by noise. Therefore, the coherent motion is decreased much more for large p. The result of Figs. 1(a) and 1(b) corresponds to the study of Ref. [18]. Figures 1(c) and 1(d) show the results for the cases of non-zero λ. The overall tendency does not change except that the coherent motion is depressed when compared to the case with $\lambda=0$. Generally the correlation of noise makes the firing events of neurons correlated so the effect of mutual excitation is diminished. As a result a coherent motion is depressed [17,23]. Interestingly, we found that when $p=0$ and $\lambda=2$, D_{opt} has a relatively low value even for a large coupling strength. When a neuron is excited by a noise, its neighboring neurons are excited by the coherent noise most probably. Consequently, spatially correlated noise enables neurons to generate firing events in the smaller noise intensity even for a relatively strong coupling. Therefore the coherent motion of the system is not drastically depressed with the correlated noise in spite of strong coupling. However, because the correlation of noise disappears mostly again for p near 1, D_{opt}s return to those values of the case for $p=1$ and $\lambda=0$.

We then systematically examined the maximal coherence factor S_m as a function of network randomness p for different values of λ (see Fig. 2). We fix coupling strengths at five values; very weak ($g\approx 10^{-2.5}$), rather weak ($g\approx 10^{-2.25}$), optimal ($g\approx 10^{-1.75}$), rather strong ($g\approx 10^{-1.25}$), and very strong ($g\approx 10^{-0.75}$).

Some interesting features are found in a certain region of

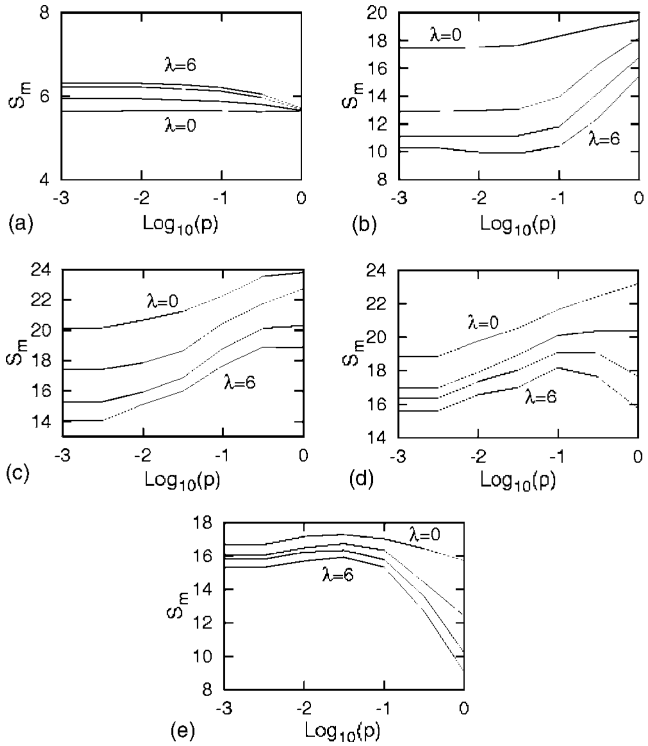

FIG. 2. The maximal coherence factor S_m as a function of network randomness p for different values of λ, when coupling strength is (a) very weak $g=10^{-2.5}$, (b) rather weak $g=10^{-2.25}$, (c) optimal $g=10^{-1.75}$, (d) rather strong $g=10^{-1.25}$, and (e) very strong $g=10^{-0.75}$. λ is varied from λ=0 (local noise) to λ=6 with a step 2. S_m curve goes down with increasing λ except in (a).

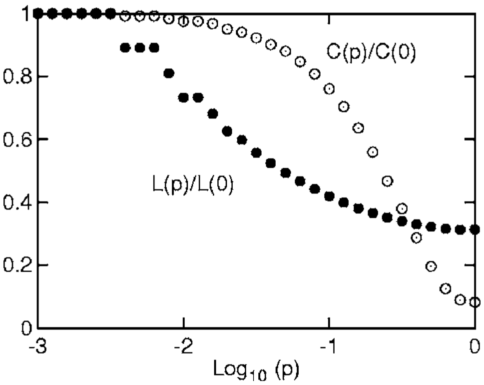

FIG. 3. Clustering coefficient C (open circle) and characteristic path length L (solid circle) as a function of p for Watts-Strogatz small-world networks with $N=101$ and $k=6$. They are normalized by each value at $p=0$.

parameter space. If the coupling strength is very weak and also if λ is zero, each element behaves independently. Therefore changing network topology does not influence the coherence of the systems. As λ increases from zero, S_m curves are elevated as increasing λ, although the elevation is very slight [see Fig. 2(a)]. The spatially correlated noise assists partially synchronized excitation among the neighboring neurons even for a very weak coupling. This synchronization is induced by coherent noise not by coupling. Hence, contrary to an expectation from the existing studies [17,23], the spatial correlation of noise enhances a coherent motion. However, as p increases, the effect of the correlated noise disappears and the partial synchronization vanishes too. Therefore, for each value of λ, S_m decreases with increasing p and finally reaches the value at λ=0. For other values of coupling strength, however, coherence deteriorates in general by the spatial correlation of noise.

In the case of rather weak coupling the effect of long-range connections is clearly observed. When λ is large for this case, we can define the transition point of p around $p_c=0.1$ above which S_m grows significantly. In Fig. 3 the clustering coefficient C remains practically unchanged (clustered structure mostly remains) for $p<p_c$ while the characteristic path length L drops sufficiently. As p passes over p_c, C drops rapidly while L rarely changes. It indicates that introducing a few long-range connections is enough to decrease L sufficiently and additional long-range connections for $p>p_c$ affect only to fracture the clustered structure. The correlated noise effectively reduces the strength of coupling term, because the difference of fluctuation of system variable between coupled neurons is small. Therefore, it is more difficult to enhance coherence of the system by a few long-range connections at larger λ. As p is increased beyond p_c, the cluster begins to be fractured and then the effect of noise correlation vanishes considerably. Hence, the effect of coupling term is recovered at large p. This makes S_m, which is definitely dropped due to the correlated noise at small p, grow rapidly as increasing long-range connections at $p>p_c$. For a larger coupling strength, the change of connecting topology becomes more crucial. By every rewiring to the long-range connections the coherent motion is steadily enhanced as p increases for entire λ [see Fig. 2(c)]. This ascending behavior of S_m due to a few long-range connections is maintained to a rather strong coupling [see Fig. 2(d)].

We observe another peculiar behavior of S_m. It has an obvious maximum value at an intermediate p around p_c for a rather strong coupling when λ>0 [see Fig. 2(d)]. For this to make sense, we need to follow the behavior of D_{opt} for each situation. When λ=0 the optimal noise intensity D_{opt} corresponding to S_m rarely changes for the entire range of p [see Fig. 4(a)]. In case of λ>0, D_{opt} remains constant at lower level than in case of λ=0 until $p\approx p_c$ and after that point gradually chases the value of λ=0. The variation of D_{opt} according to increasing p for λ>0 clearly appears for only strong coupling. With these results, we can make sure that the correlated noise enables resonance curve to occur in smaller noise level especially for strong coupling. The effect

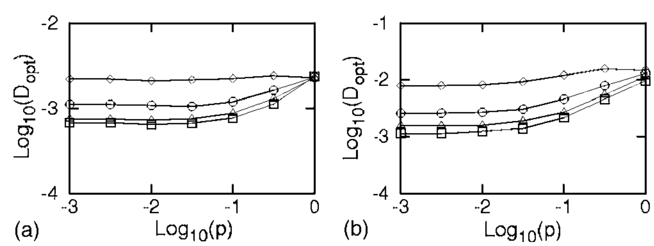

FIG. 4. Optimal noise intensity D_{pot} as a function of p for various λ including 0 (diamond), 2 (circle), 4 (triangle), and 6 (rectangle). (a) $g=10^{-1.25}$ and (b) $g=10^{-0.75}$.

of correlation of noise between inter-neurons drastically vanishes as clustered structure is fractured. As p increases further from p_c for $\lambda > 0$, resonance curve rapidly shifts to higher noises. This is why the value of S_m does not increase and even decreases after $p > p_c$.

Finally for the very strong coupling [see Fig. 2(e)], S_m changes a little until $p = p_c$ then decreases substantially for $p > p_c$. This depression is more distinct as λ increases. The behavior of D_{opt} as a function of p for various λ is very similar to the case of rather strong coupling as seen in Fig. 4(b). S_m should decrease much as λ increases when p is near zero. However, the decrease is not big owing to the effect of the reduced D_{opt}. As a whole, for $p > p_c$, the movement of D_{opt} toward larger value and a number of long-range connections with very strong coupling drastically reduce the temporal coherence of the system.

In summary, we have investigated the effect of spatially correlated noise (correlation length λ) in the presence of various connecting topology (network randomness parameter $0 \leqslant p \leqslant 1$) of FHN neural network. This study reproduces most of the general feature of AESR and AECR in the entire range of λ and p. In addition, we could get some novel features for $\lambda > 0$. When the coupling is rather strong, an optimum value $p \approx p_c$ clearly emerges where a maximal coherence resonance appears. For p increasing from 0 to p_c, the maximal coherence factor rarely changes but, beyond $p \approx p_c$, it grows dramatically either up for a weak coupling or down for a strong coupling. For p beyond p_c, it is observed that the clustered structures of the neurons are mostly fractured out due to many long-range connections and, as a result, the noise correlation of interneurons simultaneously is diminished quickly. It is believed to be the reason for such an abrupt change of S_m near $p \approx p_c$. These results show that the spatially correlated noise enhances the role of the clustered structure to the system. Therefore, the effect of a few long-range connections is ignored by the enhanced clusters of the neurons for a weak as well as for a strong coupling. Nevertheless, for an optimal and for a rather strong coupling, coherence resonance is still considerably enhanced by a small portion of long-range connections.

We would like to thank O. Y. Kwon for the computer resource for the numerical computations. Helpful discussions with Pan-Jun Kim are acknowledged.

[1] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[2] K. Wiesenfeld and F. Jaramillo, Chaos **8**, 539 (1998).
[3] B. Lindner, J. García-Ojalvo, A. Neiman, and L. Schimansky-Geier, Phys. Rep. **392**, 321 (2004).
[4] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, 453 (1981).
[5] D. F. Russell, L. A. Wilkens, and F. Moss, Nature (London) **402**, 291 (1999).
[6] Z.-C. Long, F. Shao, Y.-P. Zhang and You-Guo Qin, Phys. Lett. A **323**, 434 (2004).
[7] Hu Gang, T. Ditzinger, C. Z. Ning, and H. Haken, Phys. Rev. Lett. **71**, 807 (1993).
[8] A. S. Pikovsky and J. Kurths, Phys. Rev. Lett. **78**, 775 (1997).
[9] A. Longtin, Phys. Rev. E **55**, 868 (1997).
[10] S.-G. Lee, A. Neiman, and S. Kim, Phys. Rev. E **57**, 3292 (1998).
[11] A. Neiman, L. Schimansky-Geier, A. Cornell-Bell, and F. Moss, Phys. Rev. Lett. **83**, 4896 (1999).
[12] P. Jung and G. Mayer-Kress, Phys. Rev. Lett. **74**, 2130 (1995).
[13] S. Kádár, J. Wang, and K. Showalter, Nature (London) **391**, 770 (1998).
[14] J. F. Lindner, S. Chandramouli, A. R. Bulsara, M. Löcher, and W. L. Ditto, Phys. Rev. Lett. **81**, 5048 (1998).
[15] J. F. Lindner, B. K. Meadows, W. L. Ditto, M. E. Inchiosa, and A. R. Bulsara, Phys. Rev. Lett. **75**, 3 (1995).
[16] B. Hu and C. Zhou, Phys. Rev. E **61**, R1001 (2000).
[17] C. Zhou, J. Kurths, and B. Hu, Phys. Rev. Lett. **87**, 098101 (2001).
[18] Z. Gao, B. Hu, and G. Hu, Phys. Rev. E **65**, 016209 (2001).
[19] O. Kwon and H.-T. Moon, Phys. Lett. A **298**, 319 (2002).
[20] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[21] G. Buzsáki, C. Geisler, D. A. Henze, and X.-J. Wang, Trends Neurosci. **27**, 186 (2004).
[22] C. Hauptmann, F. Kaiser, and C. Eichwald, Int. J. Bifurcation Chaos Appl. Sci. Eng. **9**, 1159 (1999).
[23] S. Wang, F. Liu, W. Wang, and Y. Yu, Phys. Rev. E **69**, 011909 (2004).
[24] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[25] A. C. Scott, Rev. Mod. Phys. **47**, 487 (1975).
[26] R. F. Fox, I. R. Gatland, R. Roy, and G. Vemuri, Phys. Rev. A **38**, 5938 (1988).